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Elementary School Students' Ways of Thinking in Geometry Through the Lens of Geometric Habits of Mind

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Abstract

Geometric habits of mind include several ways of thinking that can be used in the learning and teaching of geometry as well as the reasoning ability in the process of solving the geometry problem. Based on this framework, revealing students' geometric reasoning process can shed light to researchers and teachers in the selection and design of methods and materials. This study is aimed to investigate the geometric problem-solving processes of elementary school students based on the GHoM. In this qualitative study case study design was adopted. The participants of the study consisted of six elementary school students. In the interviews the students can easily use reasoning with the relationship GHoM in all tasks. Generalizing and investigating invariants habits can be seen in some of the tasks. However balancing exploration and reflection habits were appeared rare when compared to other GHoMs. It can be suggested that to reveal the ways of reasoning of the students, non-routine and open-ended geometry tasks similar to those designed for this research can be utilized in mathematics classes.

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INTRODUCTION

Geometry allows individuals to analyze and solve problems with different perspectives and to establish a connection between mathematics and the real life (Dursun & Çoban, 2006). As most of the simple problems that we have to solve related to geometry in many areas in daily life are based on geometric thinking, the importance of geometric thinking in identifying a solution can be better understood (Altun, 2012). Geometric thinking is a mathematical form of thinking that involves the use of shapes, geometric concepts, relationships between concepts and principles in geometry problems (van de Walle, Karp & Bay Williams, 2007).

Research on the development of geometric thinking of individuals began with the works of Piaget et al. (Piaget & Inhelder, 1956; Piaget, Inhelder & Szeminska, 1964). Afterwards, the most well-known geometric thinking model is designed by the study of Van Hiele (1986). One of the recent theoretical frameworks in this field is Geometric Habits of Mind (GHoM) framework to foster geometric thinking and reasoning. In this research, we used GHoM framework as a lens to investigate the geometric reasoning. Productive thinking ways that become prominent in supporting geometric thinking and gaining reasoning skills specific for geometry are explained by the Geometric Habits of Mind framework proposed by Driscoll, DiMatteo, Nikula and Egan (2007). In this study, our focus is on examining elementary school students' process of solving geometry problems with the lens of geometric habits of mind.

THEORETICAL FRAMEWORK

Geometric thinking could be considered as geometric thinking (Bozkurt et. al, 2022). The Geometric Habits of Mind (GHoM) framework includes various ways of reasoning supporting the application and learning of geometric concepts (Driscoll et al., 2007; Driscoll et al., 2008). In addition, its structure that includes effective ways of solving geometry problems can help teachers predict their students' thoughts in the process of solving geometry problems. The GHoM framework consists of four ways of geometric reasoning: reasoning with relationships, generalizing geometric ideas, investigating invariants, and balancing exploration and reflection (Driscoll et al., 2007). In this framework, properties of the components are explained in the light of the GHoM theoretical framework of Driscoll et al. (2008).

Reasoning with Relationships: The GHoM of reasoning with relationships is the process of establishing a relationship in a geometric structure itself or between more than one geometric structure and determining the situations in which it can be used in solving geometry problems (Driscoll et al., 2007). An individual with this habit can identify similar and divergent aspects of geometric shapes. An example of this GHoM is shown by Driscoll et al. (2008) as like ‘relating two right triangles by the Pythagorean relationship a2+b2=c2’.

Generalizing Geometric Ideas: Generalization is the formation of a general rule based on the accuracy of a particular case (Cuoco, Goldenberg & Mark, 1996; Goldenberg, 1996). The generalization process, on the other hand, is to make predictions for the situations “often” or “always”, and control these estimates and discuss the results identified (Driscoll et al., 2008). An individual who has this habit can make inquiries about the conditions under which a feature of the geometric structure in the problem can be generalized. As an example of this GHoM, Driscoll et al. (2008) highlights students’ generalization as ‘‘If you double the size of all the sides of any polygon, you quadruple the area’’.

Investigating Invariants: This GHoM is to examine the properties which can be changed by a transformation (e.g., reflection, rotation, dissection) and the invariants which stands unchanged when others vary (Driscoll et al., 2008). An individual who has the habit of investigating invariants can determine the properties that change and remain the same when the geometric shape is manipulated or a transformation is applied to the shape. A sample of this GHoM is illustrated by

Driscoll et al. (2008) as like ‘‘Here is a triangle with perimeter 12 and area 6, and another triangle with perimeter 12 and area 4. There must be a triangle with perimeter 12 and area 5 somewhere in between.’’

Balancing Exploration and Reflection: This GHoM involves looking at the problem from different perspective and regularly going back and evaluating the situation. The questions “If so...” and ''What have I learned by doing so?'' are the indicators of this habit (Driscoll et al., 2007; Driscoll et al., 2008). An individual with the habit of balancing exploration and reflection can draw intuitively or by guessing, make changes on the shape, and make discoveries. S/he determines the intermediate steps by returning to the main problem at each step and explaining what the final situation looks like. Driscoll et al. (2008) emphasizes balancing exploration and reflection GHoM with a students’ expression like ‘‘We know how to make a rectangle from a parallelogram, so if we can make a parallelogram out of this figure, we’ll have it.’’.

The research based on the GHoM theoretical framework, is relatively recent (Driscoll et al., 2008; Erşen, 2017, 2018; Koç & Bozkurt, 2012; Köse & Tanışlı, 2013, 2014; Özen & Köse, 2013, 2014a, 2014b, 2015; Özüm-Bülbül, 2016; Özüm-Bülbül & Güven, 2019a, 2019b; Tolga, 2017; Tolga & Cantürk-Günhan, 2019; Uygan, 2016; Yalçın, 2018). Starting with the study by Driscoll et al. (2007), seminars designed to influence perceptions of teachers and middle school students about their geometric thinking were given and the framework was designed. Research on students’ geometric habits of mind framework mainly focused on the determination of GHoM’s of students or the effectiveness of the learning environments (such as Geogebra, problem-solving or origami based) that fosters students’ GHoMs (Gür, 2020; Gürbüz, Ağsu & Güler, 2018; Özüm-Bülbül, 2016; Uygan, 2016). Some of these stated that GHoMs may be fostered by geometric thinking activities (Erşen, 2018; Gür, 2020; Özen, 2015). According to them, these activities may help students to make the properties of geometric concepts more visible and make them understand the concepts that they do not understand easily. Also, research suggested that students may be introduced with problems that include multiple solutions and enhance their thinking and by this way make them flexible work while reasoning (Tolga & Cantürk-Günhan, 2020; Köse & Tanışlı, 2014).

While some researchers revealed the importance of GHoMs on geometric thinking, some others emphasized that the problems that individuals encountered in the course of geometric reasoning. Studies expressed that elementary and high school students might have problems with geometric reasoning while solving geometry problems (Tolga & Cantürk-Günhan, 2020a; 2020b). They revealed that students can easily solve procedural problems however they have trouble while they need a generalization or exploration (Tolga & Cantürk-Günhan, 2020b). Accordingly, preservice teachers encounter some problems on thinking geometrically (Koç & Bozkurt, 2012; Köse & Tanışlı, 2014; Özen, 2015). Some studies shown that GHoMs -especially reasoning with relationships and balancing exploration and reflection GHoMs- can be fostered by problem-solving activities (Özüm-Bülbül, 2016). Also, Tolga (2017) investigated in-service teachers’ GHoMs and revealed that teachers can foster more than one GHoM at the same time based on their instructional explanations.

In the light of this research, it can be stated that the GHoMs have a crucial place in geometric reasoning and in helping students to produce different ways of thinking. It is seen that the studies based on the GHoM framework are aimed at determining or fostering the GHoMs of the students. In this study we believed that investigating the geometric habits of mind of students is crucial to help teachers to plan classroom work on how to foster geometric reasoning. Literature also shows us arranging geometry problems and in-class activities based on the GHoM framework to foster the students' geometric thinking skills is considered to be appropriate in terms of gaining these ways of reasoning to the students (Driscoll et al. 2007; Driscoll et al. 2008). Research also reveals that the learning process based on the geometric habits of mind contributes to the development of individuals' geometric reasoning skills (Özüm-Bülbül, 2016; Özen, 2015; Uygan, 2016). In order to develop these environments, it could be beneficial to examine the problem-solving process of the students through the framework of GHoM and to reveal how they reason.

Therefore, this study aimed to examine elementary school students' process of solving geometry problems with the lens of geometric habits of mind. The main question that guided our study was, "How is the geometric problem-solving process of elementary school students with the lens of geometric habits of mind?". According to the main question, answers were sought for the following research questions:

How is the elementary school students' "reasoning with relationships" habit of mind take part in their geometric problem-solving process?

How is the elementary school students' "generalizing geometric ideas" habit of mind take part in their geometric problem-solving process?

How is the elementary school students' "investigating invariants" habit of mind take part in their geometric problem-solving process?

How is the elementary school students' "balancing exploration and reflection" habit of mind take part in their geometric problem-solving process?

METHOD

DESIGN OF THE STUDY

This qualitative research was designed to investigate the process of solving geometry problems of elementary school students. To investigate this process with a deeper understanding, the case study was adopted as a research design that focusing on how and why questions in the research process and where the event or phenomenon is examined in its nature (Yin, 2003).

PARTICIPANTS

The participants of the study consisted of six elementary school students from two different public schools. They have above average performance on geometry and on mathematical communication skills in mathematics class based on observations and teachers' point of view. We, as researchers, believed that the data collected from the individuals with above average performance on geometry can help us to better understand their approach to the problems. Thus, the reason that we selected these students are since we ought to investigate their understanding to the geometry problem as regards to geometric habits of mind model.

During the research process, two of the students were in 7th grade and four of them were 8th grade. Pseudonyms were given to the students as S1, S2, S3 and S4 are 8th graders and as S5 and S6 are 7th graders. Before we collected the data, all of the participants are thought the geometry field knowledge that they need to solve the problems we selected. Therefore, the participants are selected from 7th and 8th grade since their background knowledge for this study are sufficient.

DATA COLLECTION

As a data collecting instrument, four geometry tasks (Appendix 1) based on GHoM framework are designed and used in this study. The first task, "The Circles Inscribed in a Circle" aims at interpreting the relationships between perimeters (and areas) of the identical circles placed in a diameter of a bigger circle. The second one, "The circle Inscribed in a Square", is related to explicit the areas of the circles, squares and the spaces between them and find out the relationships between the shaded and unshaded areas to be formed. The third task, "Secrets of the Triangles Inscribed in Quadrilaterals", is regarding to the relationship among the area of a triangle, the square, rectangle and parallelogram. The last task, "Discoveries in the Right Triangle", is related to find out the area relationship between the geometric figures constructed on each side of the right triangle. In the last question of the task, students are asked to generalize the situation presented in each problem. Two mathematics teachers and two mathematics education researchers collaborated on the problem design. All problems were implemented as a pilot study for the validation of the problems. Afterwards, the early version of the problems revised after the feedback and final version of the problems were organized.

In data collection, clinical interview method (Clements, 2000; Ginsburg, 1997; Goldin, 2000) was used to reveal the students' ways of thinking in problem-solving process. By doing so, the interviewer asks probing questions to reveal the richness of students' thoughts and to evaluate cognitive skills (Baki, Karataş & Güven, 2002). During the interviews, the interviewer asked the questions such as, "What is asked for in this question?", "How did you determine this answer?", and "In what other ways could this rule/condition be valid?" to reveal students' ways of thinking. Within the context of this research, a total of 24 problem-solving processes were examined based on six participants each solving four tasks (6 students x 4 tasks). In this study we presented the instances which the GHoMs were clearly visible.

DATA ANALYSIS

To analyze the data, content analysis based on the theoretical framework of GHoM (Driscoll et al., 2007) were performed. Each student's problem-solving processes were examined using geometric habits of mind (reasoning with relationships, generalizing geometric ideas, investigating invariants, and balancing exploration and reflection). We develop a rubric to present which of the geometric habits of mind the questions in each task addressed. While analyzing students' interviews, we sought for indicators and evidences for each way of geometric reasoning.

For the validity of the research, expert opinion was consulted and participant confirmation was obtained. To ensure the reliability of the research, the data were first analyzed individually by two researchers, then the analysis were examined together and common themes were determined (Creswell, 2012). The coefficiency between the coders was found .89 after the analysis. After the in-depth analyses were completed, they were repeated until consensus was reached on the themes. The data analysis was continued until all of the researchers agreed on all the codes.

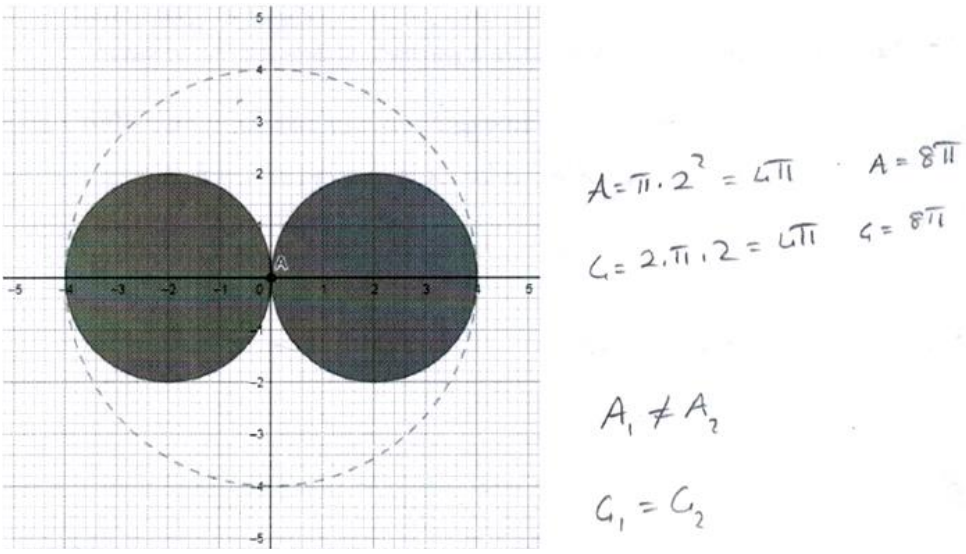
FINDINGS

In order to reveal the reasoning processes of students in geometry tasks, the findings related to the problem-solving processes of each task are presented as integrating all four activities into each of the four themes.

REASONING WITH RELATIONSHIPS

In the first task, students are expected to set up a relationship between the circumference and area of the two identical circles placed on the diameter of the large circle; and the circumference and area of the large circle. This relationship is that the sum of the circumferences of the two identical circles placed on the diameter of the large circle is equal to the circumference of the large circle, and the sum of the areas is equal to half the area of the large circle.

Figure 1. *Indicators of Reasoning with relationships GHoM of S2 on Task 1*



Out of the six participants, only S2 did not identify the expected relationship at a first sight as seen in the Figure 1. The student stated that the areas were not equal at the beginning, and afterwards, with the guidance of the researcher, she realized the ratio relationship between the areas of different circles. Her reasoning between the areas of these circles is given below.

> *-S2: There are circles. Then the area of the two circles is 8π. The circumference is 8π again … Compare with the values you found in the first question. … The areas are not equal. The area of the first is larger. The circumferences are equal.*
>
> *-Researcher (R): You said the areas are not equal or do you think there is a relationship between them?*
>
> *-S2: The area of the first question is twice the area of the second question or the area of the second question is half the area of the first question. (S2, Interview 1)*

In the third task, the relationship expected to be identified by the students is that the areas of all triangles are equal to each other wherever the position of point E in each question is located on the side AB. S1, S2 and S5 noticed this relationship by reasoning on the base and the heights of the triangles, while S3, S4 and S6 identified this relationship by giving numerical values to the side lengths of geometric figures and making various calculations.

S5's solution process for the relationship between the areas of the triangles formed by the dragging of point E can be seen below.

> *-S5: We will find the area of the other triangles by moving this (point E). We'll see what kind of relationship it is. If we drag this (point E), I think the area does not change … If we bring this (point E) here (a point on the side AB), the side EC will come there (a point on the side AB). The area will always remain the same. … (Base x Height) / 2*
>
> *-R: What can you say about the base and the height when you drag point E?*
>
> *-S5: It doesn't change. If we bring this (point E) here (a point on the side AB), the base will remain the same, the height will always remain the same. Then the area does not change. (S5, Interview 3)*

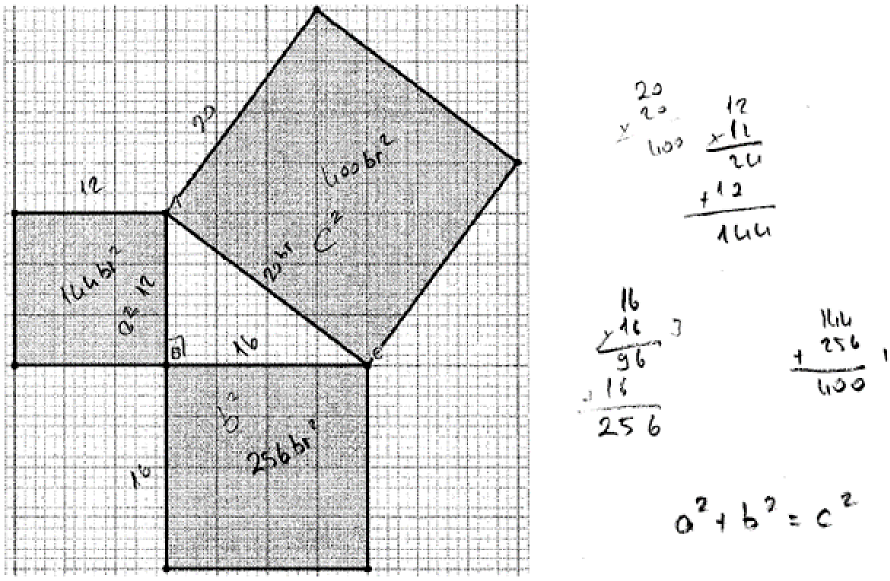
In the first question of the fourth task, students are expected to determine a relationship between the square of the lengths of the side of a right triangle and the areas of the squares constructed on the sides of triangle. The relation is that the square of any length of the side of a triangle is equal to the area of the square constructed on that side. When the findings related to the task are examined, it is striking that although Pythagorean Theorem is not in the 7th grade in the elementary school mathematics curriculum, the Grade 7 students, S5 and S6, were able to establish the expected association. S1, S3 and S4, who are 8th grade students and who know Pythagorean Theorem within the scope of elementary school mathematics curriculum, were found to have access to this association but S2 could not establish any relationship.

The solution process of S5 for this task is given below.

> *-S5: I have read somewhere that the sum of here (the length of AB) and there (the length of BC) is equal to here (the length of AC), possibly.*
>
> *-R: It's not exactly true, but you've got a clue to get into the relationship.*
>
> *-S5: According to what I said, if we add 12 to 16, it should be 28, but it is 20 units. Should we square them all? (thinking) ... I already have. The sum of 144 and 256, in other words, the sum of the areas of these two is equal to the area of that square (400). (S5, Interview 4)*

The direct quotation related to the reasoning with relationship habits of mind of the student during the solution process, S5 could not notice the desired relationship in the task at first glance but later, she expressed it by associating it with a problem situation she had previously solved (See Figure 2).

Figure 2. *Indicators of Reasoning with relationships GHoM of S5 on Task 4*



GENERALIZING GEOMETRIC IDEAS

Firstly, the problem-solving processes of the first task were examined to reveal the generalization habits of the students in the geometry tasks. In the first task, the students were expected to identify the generalization that "when n circles are placed on the diameter of any circle, the sum of the circumferences of these circles are equal to the circumference of the circle in which they are placed; the sum of the areas is equal to half of the area of the circle in which they are placed''. When the findings related to the task were examined, it was seen that all of students identified this generalization. S3's explanation about the solution process of this task "Once again, the area is 2π unit square because the area is halved each time. If we drag one more time, the perimeter will remain constant. " (S3, Task 1). As it can be understood from the explanation, the student tried to explain the generalization with her own expression by using the expressions like "If we drag one more time" and "once again".

When the third task is examined in the context of generalization, students are expected to identify the generalization that the area of the triangle DEC, from the numbered triangles, is equal to the sum of the areas of the other two triangles. When the findings related to the task were examined, it was clearly seen that S2 and S5 discovered the general expression by reasoning without requiring any processes while S1, S3, S4 and S6 found out the general expression as a result of giving numerical values to the side lengths of geometric shapes and performing numerical operations.

In addition, S1 identified this generalization at the end of the second question. In the quotation given, the reasoning process S2 performed in the solution process of the relationship between the areas of the triangles can be seen.

-S2: (Reads 2nd question of task 1) The areas (areas of the triangles 1, 2 and 3) are not equal. Their bases are not equal, but their heights are equal.

-R: Do you think there is a relationship among the areas of the triangles?

-S2: I think, the sum of the area of the triangles 1 and 3 gets the triangle 2… Because the sum of the bases on the segment AB (side AE and side EB) is equal to the base (side DC). Their heights are already equal. That's why … The sum of the bases (triangles 1 and 3) is equal to the base of the second (triangle two). (S2, Interview 3)

INVESTIGATING INVARIANTS

In the second task, students are expected to determine that the sum of the areas of the circles drawn in tangent to the sides and each other and the sum of the sum of the areas of spaces between the square and circles do not change for each question. It was observed that all of students could determine which of the other characteristics changed and which remained constant when one of the shape-specific features was changed in the next stage of the task. This shows that students have gained the habit of investigating invariants. The solution process of S1 for this task can be seen below.

In 2nd and 3rd questions, the sum of the areas of the shaded areas outside the circle inside the squares is equal to the area of the shaded area outside the circle in question 1. If this was continued, the sum of the areas of the circles would not change again. The sum of the areas of the shaded regions does not change. (S1, Task 2)

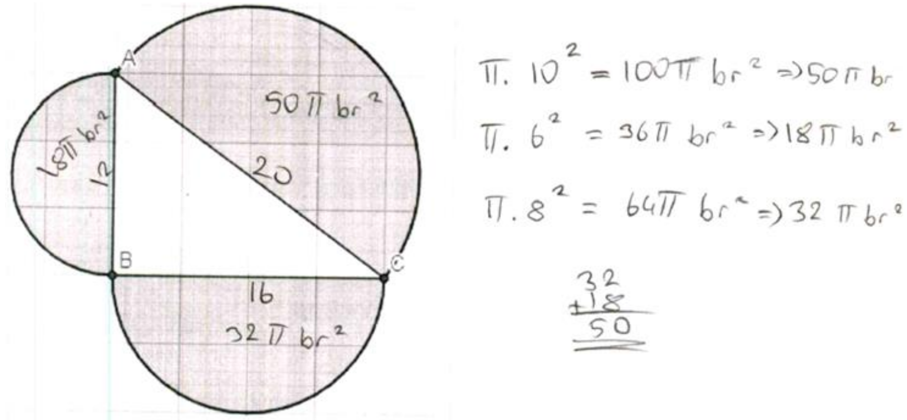
In the fourth task, the student (S6) stated that the relation that occurs when the squares are constructed on the perpendicular sides of a right triangle is also valid when the semicircles are placed. This expression was considered as an indicator of investigating invariants. According to this, only S6 mentioned that the relationship he noticed in the first stage of the problem would also be valid in other stages of the problem without making any calculation. The justification process of S6 when semicircles are constructed on the perpendicular sides of the right triangle is given below.

-S6: (Reading task 4 question 2) If the area of the small semicircle (with diameter AB) and the area of the semicircle on the side BC are added, this gives the area of the semicircle on the side AC just as above (task 4 question 1).

-R: How did we know that? Did you find out the areas? Let's first calculate the area of the semicircles.

-S6: I calculated the areas as 18π, 32π and 50π by using the formula of the area of a circle. Because of the same path with the problem above (task 4 question 1), I did the calculations (Figure 3) as 18π+32π=50π (S6, Interview 4).

Figure 3. *Indicators of Investigating Invariants GHoM of the reasoning process of S6 on Task 4*



In addition, S3's solution for the first task above is an example of investigating invariants as well as generalization.

BALANCING EXPLORATION AND REFLECTION

In the fourth task, students are expected to produce arguments on various regular/irregular polygons and explain them in formal mathematical way. In the problem-solving process S2 could not reason with any of the habits, and S1 and S5 constructed a polygon without justifying it in a formal way. However, S3, S4 and S6 produced arguments on the polygon constructed and justified their thinking mathematically. In the solution process the explanations of S1, S3, S4 and S6 are similar and as follows.

-R: What does it mean by referring "similar relationship"?

-S1: Well, it can be just like here (question 3). The sum of the areas of the small figures can be equal to the area of the large figure.

-R: Can you think of any other shapes we can draw?

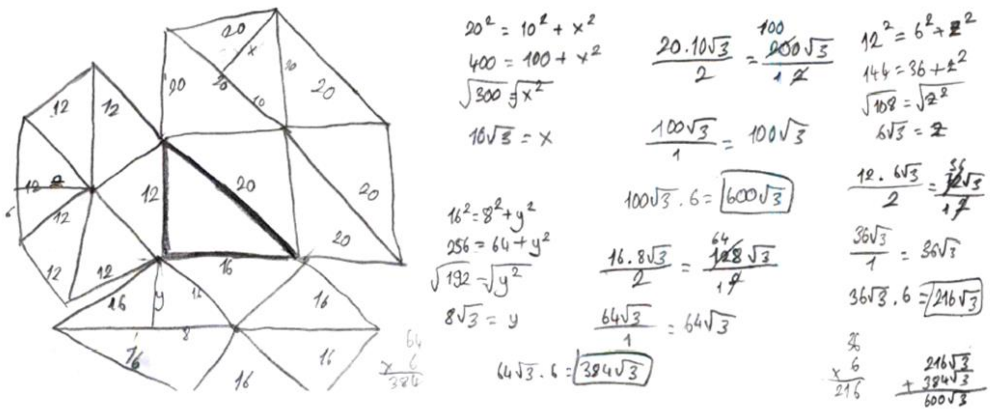
-S1: Maybe, an isosceles triangle can be drawn.

-R: Why isosceles triangle?

-S1: It's easier to find the area in the isosceles triangle. The height drawn from 3rd vertice is both median and angle bisector. For me it's easier to make an operation with this. (S1, Task 4)

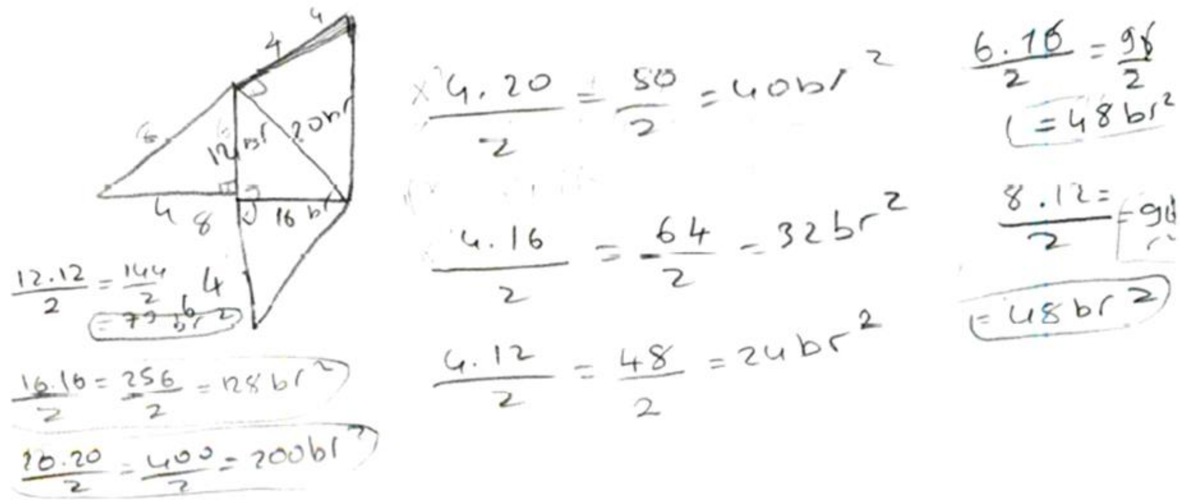
S3 also explains that the relationship determined by constructing regular hexagons on the sides of the triangle given in the task can be obtained on the shape formed by itself (Figure 4).

Figure 4. *Indicators of Balancing Exploration and Reflection GHoM of the reasoning process of S3 on Task 4*



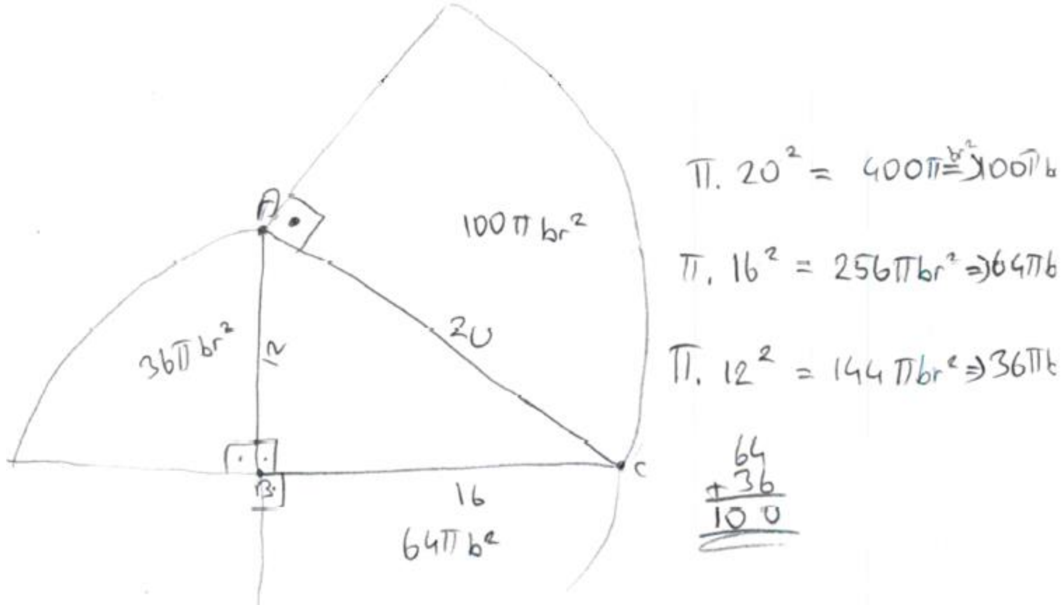
In the same task, S4 constructed the right scalene triangles in the beginning. One of the legs of these triangles are located on each side of the right triangle as seen on Figure 5 and the other legs have 4 centimeters length. Then, she noticed that this way of thinking did not help her to find a relationship. She tried to calculate whether any other side lengths could help her to process the relationship. In her third attempt, she constructed the isosceles right triangles located to the sides of the right triangle given in the task. She tried to explain mathematically through the mathematical operations that the relationship can be achieved with the isosceles right triangles she constructed.

Figure 5. *Indicators of Balancing Exploration and Reflection GHoM of the reasoning process of S4 on Task 4*



S6 constructed quarter circles with the radius located on each side of a triangle given in the task. Then he justified that the relationship he noticed before in the former stages of the task is valid in new case.

Figure 6. *Indicators of Balancing Exploration and Reflection GHoM of the reasoning process of S6 on Task 4*



In addition, Figure 6 (first stage of the fourth task) indicates that the 7th grade student (S5) noticed that the sum of the squares of the lengths of the legs was equal to the square of the hypotenuse although she did not learn the Pythagorean Theorem. This indicated that she reasoned with exploration and reflection habit besides relationships.

DISCUSSION, CONCLUSION AND IMPLICATIONS

In this study, the reasoning ways of 7th and 8th grade students in the geometry problem-solving process were examined based on the Geometric Habits of Mind framework. One of our most important results was that the majority of students were reasoning with relationships, generalizing geometric ideas and investigating invariants at an advanced or adequate level. However, in our context, the balancing exploration and reflection habit of mind was the least common habit among the participants while they were reasoning with these tasks. Similar situation emerged in some research results (Koç & Bozkurt, 2012; Köse & Tanışlı, 2014; Özen, 2015) and also some studies (Lim & Selden, 2009; Köse & Tanışlı, 2014) suggest that students should adopt these habits to increase their geometry performance.

While we consider each GHOM revealed in students' reasoning process, we can clearly focus students' geometric thinking and how students reason. For the reasoning with relationship GHoM, we found that students were able to reason at the desired level except for one of the students (S2). Also, the relationships within and between geometry tasks were noticed by the students. In particular, the fact that the relationships in the task related to Pythagorean Theorem was explored by 7th grade students (S5 and S6), who did not cover the subject in their school mathematics courses, suggests that GHoMs developed in some students independently of the outcomes in the curriculum and through the reasoning they used in the tasks they solve.

As for the generalizing geometric ideas GHoM, students are expected to be able to determine a pattern or notice a rule based on the cases presented in the tasks. Also, students are generalizing geometric ideas when they seek solutions to recognize that the given conditions work for an infinite set by considering only a discrete set and to situate a problem or rules in broader contexts (Driscoll et al., 2007). The vast majority (five out of six students) of students predicted the consequences they would face in the later stages of the tasks by following the right reasoning processes and succeeded in building a general rule. However, one student (S2), who partially adopted the reasoning with relationships GHoM, had also difficulty in generalizing geometric ideas depending on this. This may due to the fact that the reasoning with relationships GHoM forms the basis of other habits of mind in mathematics and especially in geometry. Although it is stated that the GHOM framework (Driscoll et al. 2007), are not hierarchical as in the van Hiele (1986) model, it was seen in our study that generalizing geometric ideas GHoM cannot exist without reasoning with relationships GHoM. However, the opposite of this situation is not valid; in other words, students have reasoning with relationships GHoM where there is no generalization.

As to the investigating invariants GHoM, the majority of students were able to determine which of the other characteristics changed or which of them remained constant when one particular characteristic of the given geometric structure was changed in the next stage of the task. This situation was accepted as an indicator that students reasoned with the investigating invariants GHoM. In the first task of this study, as the problems of the task progressed, all of the students were able to determine that the circumference remained constant. In addition to this, S2 is the one who was able to determine that areas came to the half with a guidance of a researcher. All of students but S2 were able to determine that areas came to the half as a result of their own reasoning processes. In the second task, the conclusion that the sum of the shaded areas between the squares and circles remained constant for each problem was noticed by all of students. Especially, S1's way of justifying her explanation based on the reasoning (the sum of the shaded areas between the square and the circles in the second question would not be equal to the result) made in the 3rd question was remarkable. When she was asked about the reason for her justification in the second question, she said "Probably, it was because that it was my first time because I got used to it here (Task 3). It was better.". Their difficulty in geometric reasoning might be a result that they are not sufficiently faced with such tasks that foster geometric thinking in their school mathematics curriculum.

In the third problem, some students (S3, S4 and S6) were able to determine the conclusion that the areas are equal to each other because the base and heights of the triangles formed by moving the E point through the calculations they made. However, some students (S1, S2 and S5) determined the same conclusion without using calculations. Finally, at each stage of the fourth problem, all of students except S2 determined the generalization that the sum of the areas of the geometric figure placed on the perpendicular sides of the right triangle was equal to the area of the geometric figure placed on the hypotenuse. As mentioned with the reasoning with relationships GHoM here, it is remarkable that 7th grade students (S5 and S6) were able to conclude with this situation without any information about Pythagorean Theorem from their school mathematics courses.

We mainly noticed students' balancing exploration and reflection GHoMs at the fourth task. In the final stage of the task, with other polygons were constructed on the sides of the right triangle, students were asked to identify the desired relationship. Three of the students formed hypotheses about the solution by considering similar situations in the previous stages and would rather explain these in a mathematically convincing way. S3's regular hexagon construction, S4's isosceles right triangle construction, and S6's quarter-circle construction on the sides of the right triangle are indicators that they have balancing exploration and reflection GHoM. Two students (S1 and S5) expressed their arguments verbally and stated that it was difficult and impossible to explain mathematically without drawing auxiliary lines or any modifications.

The reason for this tendency among students might be current education in schools is not regulated to support and evolve such mental processes. In the same task, one student (S2) could not form any hypothesis about the whole task. When the reasoning process of all of the students are evaluated together, the balancing exploration and reflection is the least adopted GHoM within all of others. Similarly, this situation is in line with some research results (Koç & Bozkurt, 2012; Köse & Tanışlı, 2014; Özen, 2015).

As a result, our study showed us the lack of GHoMs in students' reasoning and noticed us the importance of mathematics courses need to include activities that promote them. To enrich students' reasoning processes in geometry tasks, the task designed for this research, or the non-routine and open-ended geometry tasks developed by Driscoll et al. (2007, 2008) should be used in middle school mathematics.

The findings obtained in this study are limited to the process of solving four open-ended geometry tasks developed by the researchers. In the forthcoming research, the studies that will be planned to determine the GHoMs of the students should be enriched by long-term classroom observations. Although the problem-solving processes of the students are examined in a paper and pencil environment, the use of various materials and dynamic geometry software in determining or fostering the GHoMs of students can enrich the process and reveal the perspectives that the students cannot gain in a traditional setting.

AUTHOR CONTRIBUTION

- First author (corresponding author) has made substantial contributions to conception and design, or acquisition of data, or analysis and interpretation of data.
-The second author has been involved writing and drafting the manuscript and revising it critically for important intellectual content.
- The third author has been involved in collecting and analyzing data.

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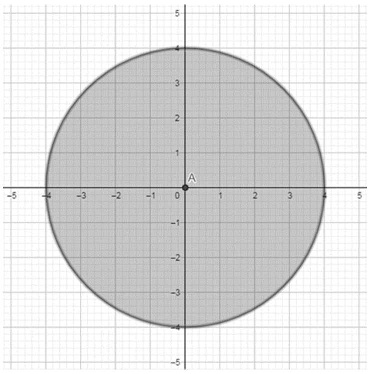
APPENDIX

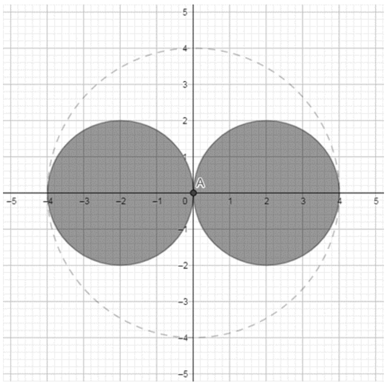
Problems

Task 1. The Circles Inscribed in a Circle

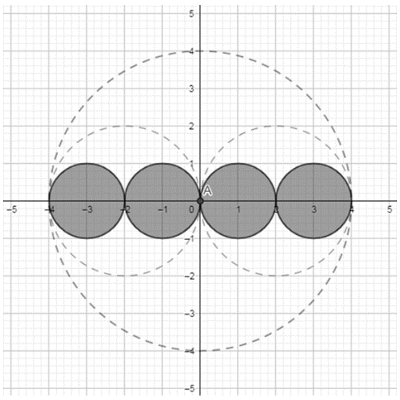
1) Find the area and circumference of the circle given in the coordinate plane.



2) Find the area and the circumference of the shaded area consisting of circles drawn tangent to each other in the coordinate plane. Compare the results with the values in the first question.



3) Find the area and the circumference of the shaded area consisting of circles drawn tangent to each other in the coordinate plane. Compare the values you found with the findings obtained from the first and second questions. Describe changing or unchanging features and situations.



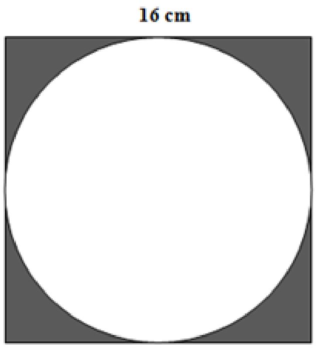
4) Can we say a general rule considering all these situations? Why?

Task 2. The Circle Inscribed in a Square

1) In Figure, a circle is drawn tangent to the sides of the square.

a) Find the area of the given circle.

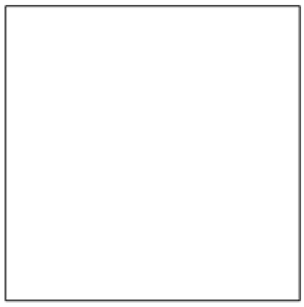
b) Find the sum of the areas of the spaces (shaded areas) between the square and the circle.



2) Divide the given square into 4 equal squares and draw circles inside each frame so that they are tangent to the sides as in figure.

a) Find the sum of areas of the circles formed.

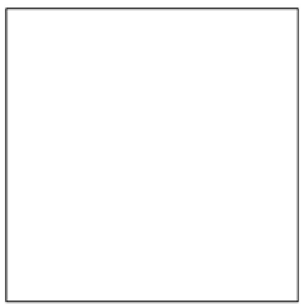
b) Find the sum of the areas of the spaces between the squares and the circles.



3) Take the second question one step further.

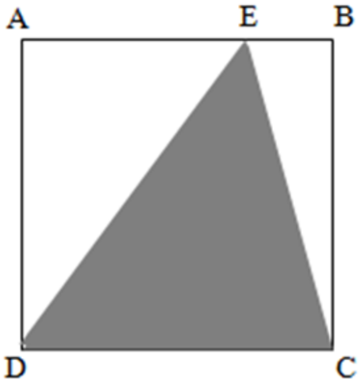
a) Find the sum of the areas of the circles.

b) Find the sum of the areas of spaces between the squares and the circles.

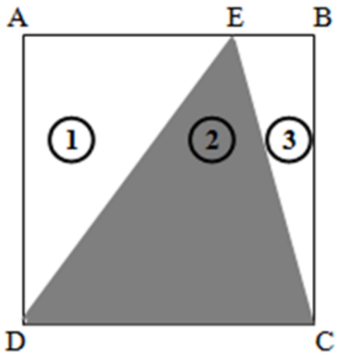


4) When you compare the values you found in the questions, what are the characteristics and situations that change or not? What can you say about the results that can be achieved when this process is continued? Please explain.

Task 3. Secrets of the Triangles Inscribed in Quadrilaterals

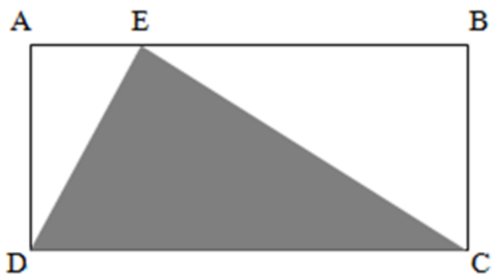


1.a. In the figure, a square and a triangle drawn inside the square are given. Since the vertex of the triangle (E) is a moving point on the side, what is the relationship between the areas of different triangles to be formed by the movement of the point E?

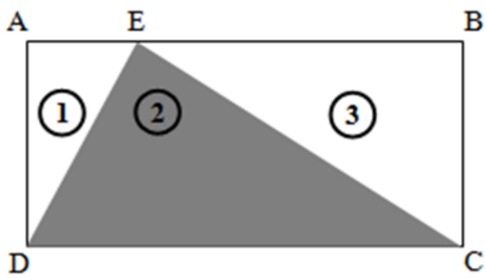


1. b. What is the relationship between the areas of triangles 1, 2 and 3?

1.c. Is there a relationship between the area of the square and the areas of the triangles? Please explain.

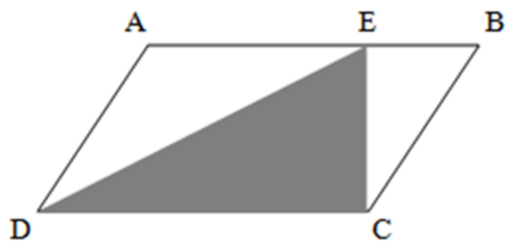


2.a. A rectangle and a triangle drawn inside that rectangle is shown in the figure. Since the vertex of the triangle (E) is a moving point on the side, what is the relationship between the areas of the different triangles to be formed with the movement of the point E?

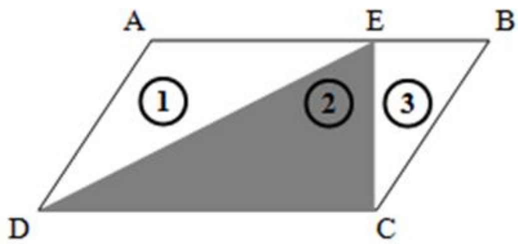


2.b. What is the relationship between the areas of triangles 1, 2 and 3?

2.c. Is there a relationship between the area of the rectangle and the areas of the triangles? Please explain.



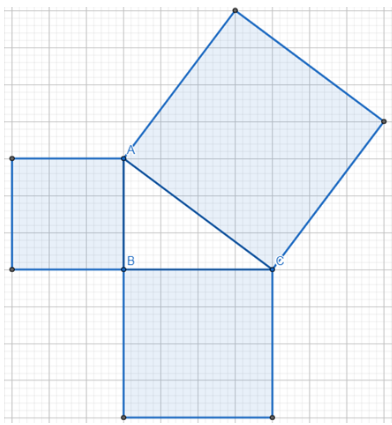
3.a. A parallelogram and a triangle drawn into this parallelogram are shown in the figure. Since the vertex of the triangle (E) is a moving point on the side, what is the relationship between areas of different triangles with the movement of the point E?



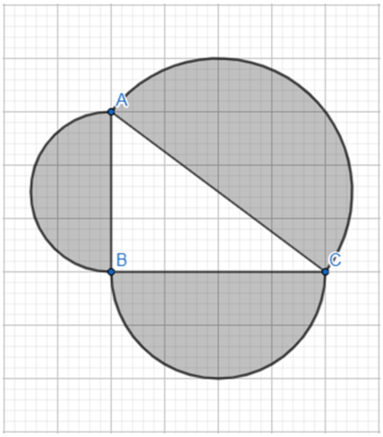
3.b. What is the relationship between the areas of triangles 1, 2 and 3?

3.c. Is there a relationship between the area of the parallelogram and the areas of the triangles? Please explain.

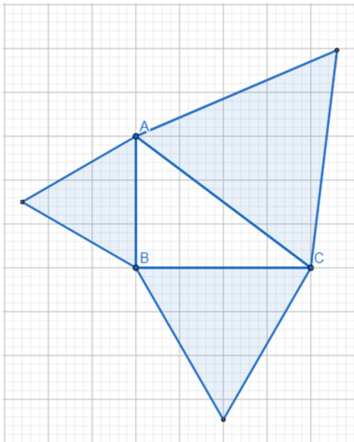
Task 4. Discoveries in the Right Triangle



1) Find the areas of the squares constructed on the sides of the right triangle ABC with side lengths 12, 16 and 20 units. Is there a relationship between the values you find?



2) Semicircles are constructed on the sides of the right triangle ABC. Is there a relationship between the areas of semicircles? What is the reason for that?



3) Equilateral triangles are constructed on the sides of the triangle ABC. Is there a relationship between the areas of these triangles? Why?

4) Explain which other polygons can be constructed on the sides to explore a similar relationship.