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Investigation of Classroom Practices of Middle School Mathematics Teachers in the Context of Geometric Reasoning Processes

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Abstract

Cognitive processes and cognitive apprehensions known as geometric reasoning processes play a significant role in enabling students to make geometrical deductions and develop their spatial skills, geometrical skills, imaginations, and geometrical intuitions through geometrical properties, to discover the transformations between geometrical models and to establish a bond between the concepts. This study examined classroom practices of middle school mathematics teachers in the context of cognitive processes and cognitive apprehensions in geometry teaching. In this context, using the descriptive survey model, three classroom practices of middle school mathematics teachers with three different levels were examined. The descriptive analysis method was used to analyze these data. According to the findings obtained from the data, it was observed that geometric reasoning processes differed in each teacher's classroom practices. It has been observed that the most common dimensions of geometric reasoning in courses with geometry content are visualization and reasoning. It was further observed that cognitive apprehensions were involved simultaneously with the cognitive processes. In this respect, it was concluded that the processes are interactive with each other.

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INTRODUCTION

The reasoning is the process of conducting to query using the questions "Why?" and "How?" to concern a problem or an issue that one seeks a solution for and to give meaning to it (Dyer and Sherin, 2016; Lu and Richardson, 2018). If the question, which a solution is sought for, is of mathematical content, it can be named "mathematical reasoning"; yet when made even more specific, it can be named "geometrical reasoning" if it has contents regarding geometry. Geometrical reasoning can also be defined as making a logical deduction by contemplating any given geometrical problem, considering all probable factors (Duval, 1998). The structure of the geometrical reasoning is characterized by the quality of the correlation between the concept and shape.

The effectiveness of reasoning in learning geometry was outlined in many studies in the relevant literature (Clements & Battista, 1992; Duval, 1998; Fuys, Geddes & Tischler, 1988; Jones, 1998; Mabotja, Chuene, Maoto & Kibirige, 2018). In addition, it is possible to come across some studies showing that reasoning skill improves problem-solving skills as well (Barker, 2003; Briscoe & Stout, 2001; Kramarski & Mizrachi, 2004; Lithner, 2000; Schoenfeld, 1985; NCTM, 1989; 2000; Santos-Trigo, 2014). These studies stated that academic achievement and problem-solving success are highly correlated with reasoning, and students with better reasoning skills are more successful in learning mathematics than others. It has been stipulated that reasoning skill is among the fundamental skills concerning the learning and use of mathematics, which is why the requirement of preparing learning environments aimed to improve reasoning was put forward (NCTM, 2014). In this context, the teachers' intra-classroom practices gain significance in improving the students' reasoning skills (NCTM, 2000). This study aims to determine the cognitive processes and cognitive apprehensions of the geometric reasoning process and the interaction in geometry-based mathematics courses. The study aims to explore cognitive processes and cognitive apprehensions and the relationship between these processes while being taught figurative and conceptual information in classroom practices of middle school mathematics teachers' mathematics courses. In this context, answers were sought for the following questions:

- What are the aspects of cognitive processes and cognitive apprehensions among the geometric reasoning processes that highlight in classroom practices of middle school mathematics teachers?
- What is the relationship between aspects of cognitive processes and cognitive apprehensions in classroom practices of middle school mathematics teachers?

THEORETICAL FRAMEWORK

Mathematics is attempting to understand everyday situations from a quantitative standpoint by filtering them through reasoning (Lithner, 2008). Fischbein and Schnarch (1997) state that developing the students' reasoning skills facilitates learning mathematical subjects. Therefore, it has become necessary that studies be carried out to improve the reasoning skills in every field of mathematics. It is likely to come across many approaches in the literature concerning the nature of and how to improve the process of reasoning in geometry, which is one of these sub-fields of learning (Duval, 1995; Fishbein, 1993; Piaget & Inhelder, 1967; Van Hiele,1957)**.** Although the van Hiele model is the most well-known among these approaches, the Figural Concept theory of Fischbein and the Cognitive model of Duval are other approaches held in the literature to analyze the process of reasoning in geometry (Jones, 1998).

In a study they carried out in 1957, the van Hiele couple drew attention to the quality of teaching levels of geometrical thinking and the transition from one level to another (Fuys, Geddes & Tischler, 1988). As seen in Piaget's approach, van Hiele analyzed the thinking processes through the developmental approach (Battista & Clement, 1995). This model approaches geometrical thinking at five levels: visualization, analysis, informal deductive, formal deductive, and rigor. When the studies in the field of geometrical reasoning (thinking) are analyzed, it is possible to come across quite a many studies made on Van Hiele's geometrical thinking levels (Burger & Shaughnessy, 1986; Fuys, Geddes & Tischler, 1988; Mason, 1998; Solaiman, Magno & Aman, 2017; Usiskin, 1982).

Fishbein's figural concept model is another prominent theory within the context of geometrical education. According to Fishbein (1993), geometrical reasoning correlates the geometrical concept and geometric shape. As per this opinion, while concepts provide a mathematical foundation for our results, geometric shapes help make estimates and use intuition in reasoning. Thus, an exemplary reasoning process depends on the interaction of the knowledge of shapes and concepts.

In his study, Duval (1995) further elaborated on the figural concept model of Fishbein (1993). Duval explained the reasoning through cognitive processes and cognitive apprehensions experienced when one looks at geometric shapes. He claimed that effective geometry teaching could occur through interacting these processes with one another (1995, 1998). The cognitive processes in the model put forward by Duval consist of visualization, reasoning, and construction. Cognitive apprehensions, however, consist of perceptual apprehension, discursive apprehension, sequential perception, and operative apprehension. Visual demonstration of a situation in geometry helps carry out functions, such as a general overview of the current situation, instantaneous perceptions, and personal verification. This demonstration covers the construction of the shape via tools-instruments like concrete materials and dynamic geometry software or the features regarding the sorting of the construction process to let the change and expansion happen in the current knowledge and for the construction of a geometric shape (Torregrosa & Quesada, 2008). Duval's geometric reasoning processes have been examined in two categories: cognitive processes and cognitive apprehensions. In order to determine the emphasis on the categories determined during the in-class practices of teachers, dimensions are explained in Table 1-2, and examples suitable for these dimensions are given.

Table 1. *Cognitive Processes*

Aspects	Definition/Explanation	Examples
Visualization	This is the process by which a place is visually represented to perform functions such as a visual representation of a situation, an overview of the current situation, instantaneous perceptions, and subjective verification. These representations are themselves geometric shapes that contain mathematical properties.	*The teacher shows 3 o'clock on the wall clock and asks the students what angle between hour and minute hands is.*
Construction	This includes creating a model of any geometric shape or sequencing the construction process using tools such as a compass, ruler, and dynamic geometry software to create shapes.	*The teacher asks whether it is possible to draw a triangle, the side lengths of which are 3-4-5 units, using a ruler and a compass, and tries to create a model of this triangle in company with the students.*
Reasoning	This is the occurrence of a change and expansion in knowledge. Reasoning processes, which appear based on the features of the demonstration forms used, were divided into two: - Natural discursive process (5/A: inference from the shape) - Theoretical discursive process (5/B: definition, theorem, axiom, deduction)	*"What is called the shape formed by the two rays with common starting points ?" By asking the question, the teacher enables the students to reason using their theoretical knowledge (5/B).* *Finding the relevant angle from the figure by looking at the figure through the angle model formed by two parallel and one intersecting line shows a natural discursive process (5/A).*

As seen in Table 1, Cognitive processes were classified by Güven and Karpuz (2016) under three aspects: visualization, reasoning and construction based on Duval's model (Duval, 1998).

Table 2. *Cognitive Apprehensions*

Aspects	Definition/Explanation	Examples
Perceptual	- The initial look at the shape - Name, size, basic geometrical elements of the shape (point, straight line, triangle) - Determining the sub-shapes of the shape (determining the triangle within the rectangle)	*The teacher asks the student to determine what geometric element it is by showing a pencil with an open tip. Students say it is a ray.*
Discursive	It is the process of establishing a relationship between shape and mathematical principles (definition, theorem, axiom, etc.) in order to infer what is desired based on what is given.	*The teacher asks the students to give an example of the line segment that can be seen in their daily lives and, based on these visuals, the definition of the line segment.*
Sequential	- Creating a shape, using tools (compass, rulers) - Describing the process of construction without any tools (the aspect of construction in the cognitive processes and this aspect support one another)	*The teacher asks the students to draw a square and describes the process needed while creating the square model.*
Operative	- The endeavor to get a clue, intuition, solution, perspective - Making changes to the first shape - Drawing, erasing, adding, and displacing auxiliary straight lines - Thinking more over some parts of the shape compared to others	*The teacher writes a problem on the board; nevertheless, the students fail to solve the problem as they see an unusual shape. When the teacher turns the shape sideways and adds auxiliary straight lines, the students solve it.*

As seen in Table 2, cognitive apprehensions are classified by Guven and Karpuz (2016) as four aspects: perceptual, discursive, sequential, and operative (Duval, 1995).

Many recent studies have related to Duval's cognitive model (Kose, 2014; Ocal & Simşek, 2017; Ramatlapana & Berger, 2018; Trigueros & Martínez-Planell, 2010). These studies have been conducted at a single level over specific theory categories. For instance, in a study, Kose (2014) discussed the construction aspect of the cognitive process category. Discussing the sequential aspect of the cognitive apprehensions, Ocal and Simsek (2017) focused on the phases of generating solutions used by secondary school mathematics teachers to find a solution for geometrical construction problems, as well as their opinions on this matter. While the theoretical frameworks Fishbein and van Hiele put forward in their work provide a view from the student's point of view, the theoretical framework that Duval put forward enables us to analyze the teaching process.

The cognitive model of Duval (1995), which constitutes the theoretical framework in this study, is important because it consists of cognitive processes and cognitive apprehensions and the interaction between sub-dimensions that make up these two categories, in contrast to a hierarchy. It provides a more holistic view of the geometric reasoning process. In addition, this study is about revealing the geometry teaching process in classroom settings. It is thought that mathematics courses with the geometry content study will contribute to mathematics education in terms of developing the literature by providing the opportunity to examine cognitive and cognitive apprehensions together in the context of geometric reasoning.

METHOD

RESEARCH DESIGN

This research is a descriptive study based on a survey model. According to Karasar (2005), survey models are research approaches that aim to describe a past or present situation. This method tried to determine the teachers' reasoning processes in their classrooms' geometry applications. To make

Duval's model visible, video analysis was used in this study due to the chance it offers to discuss different teachers' geometry-oriented classes and derives rich data from them.

PARTICIPANTS

The study group consists of 5 secondary school teachers working as mathematics teachers in public schools in a province in southeast Turkey. While choosing the study group, the criterion sampling approach, one of the purposive sampling methods, was used. Criterion sampling studies situations that meet predetermined criteria (Yin, 1984). Different professional experiences of teachers, different grade levels, and gender were considered criteria. The demographic information of the teachers in the study group is given in Table 3.

Table 3. *Demographic Information of Teachers is Held in the Study Group*

Teachers	Gender	Professional seniority	Classroom level recorded on video
T1	Male	16 years	8th- grade
T2	Female	4 years	7th- grade
T3	Male	6 years	7th- grade
T4	Female	5 years	6th- grade
T5	Female	7 years	6th- grade

As seen in Table 3, three female and two male teachers are held in the study group. The service years of the participants range from 4 to 16 years. The recorded classes are chosen as two at the 6th-grade level, two at the 7th-grade level, and one at the 8th-grade level.

DATA COLLECTION

This study includes using an image-based observation technique since it allows the researcher to observe teachers' classroom practices and interpret observed practices. Fraivillig, Murphy, and Fuson (1999) define the observation technique as a data collection technique that should be preferred to investigate the behaviors in a particular environment or institution more detail. In this study, the footage was recorded by two professional cameramen working in a state university's television and cinema department. During the recording phase, while the first cameraman focused on the teacher's practices in the classroom, the second cameraman recorded the students and their breakthroughs, and interactions with each other and their teachers. The classroom observation program was planned according to the preferences of the participating teachers, and 40 minutes, a course hour with only geometry content, was recorded.

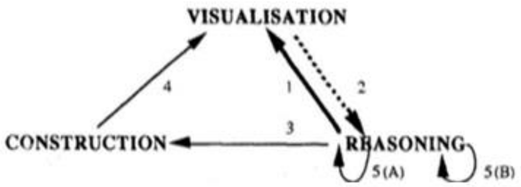
DATA ANALYSIS

A researcher first transcribed the data obtained from the images, and another checked each transcription data set for the accuracy and validity of the observed actions. After the transcription level, the data set was analyzed using the descriptive analysis method. Data are summarized and interpreted in the descriptive analysis according to predetermined codes (Patton, 2014). Dimensions of the cognitive processes and cognitive apprehensions in the geometric reasoning processes were used in the mathematics course-classroom applications with geometry content (Table 1 and Table 2).

DIALOG MAPS

The dialog maps of the classroom practices of middle school mathematics teachers were created to determine the interaction between the geometrical reasoning processes. The interaction between the cognitive processes is given in Figure 1.

Figure 1. Interaction between Kinds of Cognitive Processes (Duvall, 1998)



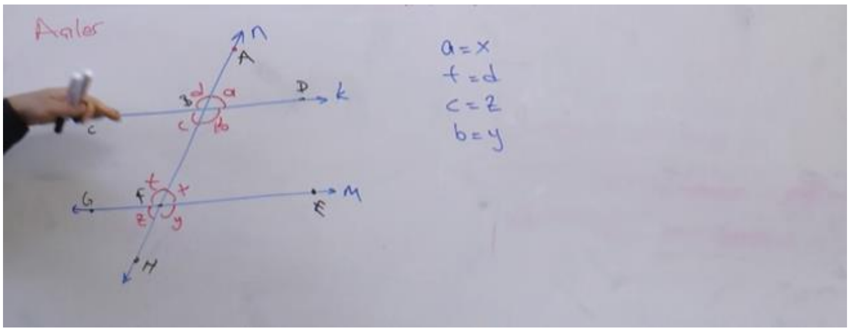
While creating the maps of the cognitive processes and cognitive apprehensions, the processes were divided into significant sections according to the beginning and end of the dialogues of each sample question, solution process, or practice.

Geometric activity chains were obtained due to the interaction between the aspects defined within the analysis framework in those sections. A sample dialog is provided to clarify further how these chains were obtained.

SAMPLE DIALOG

The sample dialogue is taken from an in-classroom activity where T3 draws a shape that consists of two lines on the board and one straight that cuts them and ends the process by determining interior angles.

Figure 2. *The Angle Model was Drawn by the Teacher.*



T: Now, when you look at it, which angles remain in between these two parallel lines (showing with his hand)? Tell me, Alperen.
S2 (Alperen): An acute angle, teacher.
T: Hmm, which angles? I want the names.
S (Alperen): x, x and b.
T: x, b. anything else? Yes, Ibrahim.
S (İbrahim): t, c
T: t and c. These four, right?
S: Yes.
T: Now, these are the interior angles. The name of the rule is interior alternate angles. If these angles are inside, which of these angles are alternates for another? (Hands risen by the class). Ahmet.
S (Ahmet): y and z are alternates, teacher.
T: These are exteriors, but they should be interiors and alternates.
T: All right, let me say it, its x and c.
S: Some students say t and b.
T: Also t and b. These should be equal. (Moving towards under the heading) which ones again?
S: x and c.
T: x and c (writes it on the board).
S: t and b.

(08.05) section 3 *T: t and b are true.*

In this course, T3 visualizes the angle formed by two parallel and one transverse line with a cutting line. Using this visualization, T3 asks the students to make an inference from the shape through reasoning. While the students are talking about the question, we see that they consider the shape and answer by interpreting the shape. This statement is a natural *discursive process* represented by 5 (A) and occurs independently. This reasoning process tries to get the person to find alternative angles within the given and through the image itself.

The direction of perception is evaluated in the construction (3) direction context. This is indicated by arrow number 3 on the dialogue map. Asking students to give the figure the basic geometric elements (angle) at first glance is evaluated simultaneously with the visualization aspect in the context of perceptual comprehension. In addition, the teacher, which allows students to make inferences by reasoning on what is given in Table 4, performs activities that stimulate discursive comprehension.

Table 4. *Dialogue Map in the Context of the Reasoning Processes of a Sample Dialog*

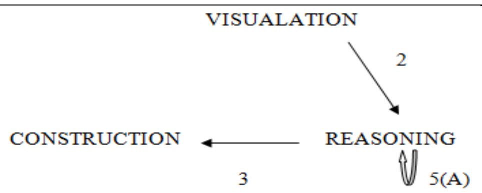
Cognitive process	Cognitive apprehensions
	perceptual - discursive – sequential

Table 4 shows that the cognitive processes were coded as 2-5(A)-3 in the sample dialog. The arrow numbered 2 in Figure 2, given for the sample dialog, expresses the unidirectional interaction between the visualization and reasoning aspects in the cognitive process category. The sample dialog coded the cognitive apprehensions as perceptual - discursive – sequential.

VALIDITY AND RELIABILITY

More than one teacher's lecture videos were examined in order to increase the validity and generalizability of the findings of the study. Within the context of the reliability of the data analysis, two separate researchers carried out coding processes through the cognitive processes and perceptual apprehension aspects over the transcribed version of the footage of a participant independently. The questions with “Consensus” and “Dissidence” were determined by comparing the responses the researchers gave based on the aspects available in the framework used within the scope of the descriptive analyses. If the researchers expressed the same aspect in the relevant section, this was considered a consensus; yet if they marked different options, this was deemed a dissidence. The “interrater reliability” was found as 83% (35/42) for the cognitive processes and 81% (22/27) for the cognitive apprehensions in the study, and this rate is deemed to be reliable in the sense of Miles and Huberman (1994). Although the rate obtained was considered reliable, the researchers gathered and discussed until they reached a consensus over the points of disagreement. Thus, the reliability of the data analysis was improved through deviant cases.

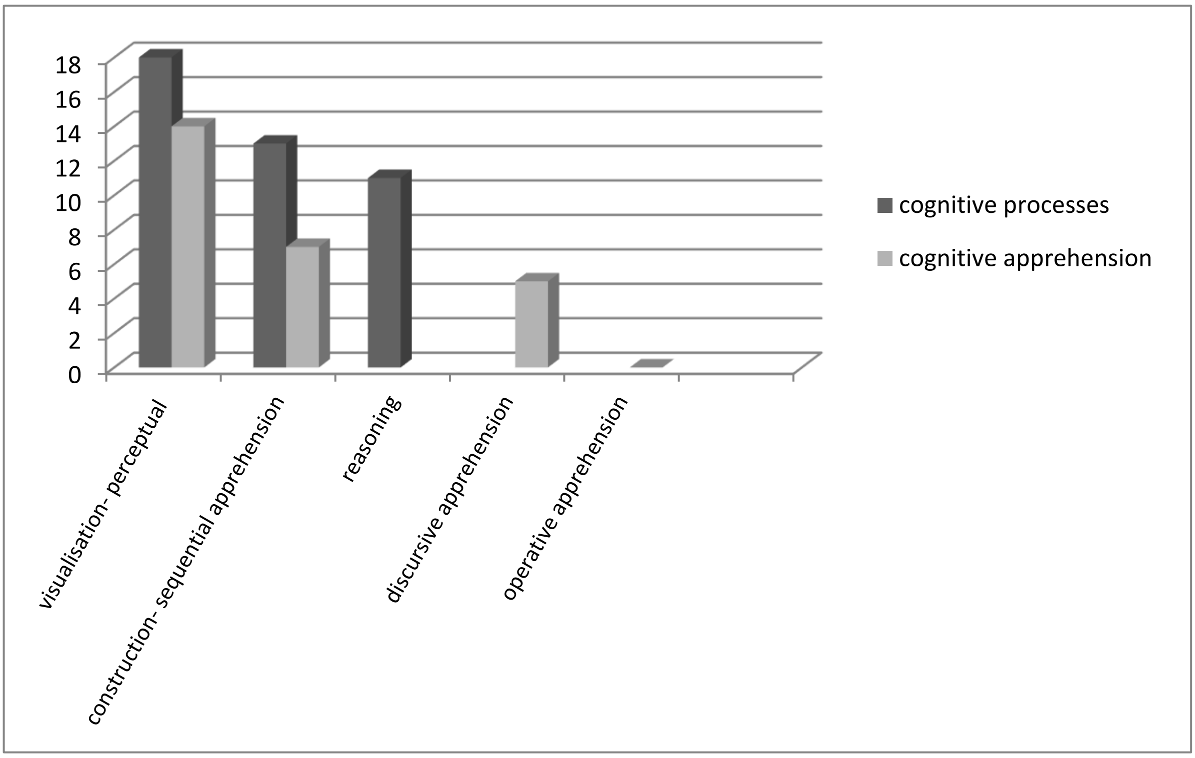
FINDINGS

The results obtained from this study are presented as the frequency of using the geometrical reasoning processes and the dialog maps in each section of the courses for each participant.

CLASSROOM PRACTICES OF T1 IN THE CONTEXT OF GEOMETRIC REASONING PROCESSES

The aspects obtained from the analysis of the geometrical reasoning process of classroom practices of T1 are given in Diagram 1.

Diagram 1. *Geometric Reasoning Processes and Frequencies in T1's Classroom Practices*



When Diagram 1 is examined, the geometry-based class of T1's classroom applications shows that these two aspects are simultaneous and support each other in the cognitive process category by evaluating them with the visualization aspect (18) and the perceptual comprehension aspect (14). These two categories are similar in numbers. These numbers point out the relations between the processes, as stated in the work of Guven and Karpuz (2016). Furthermore, the fact that evaluations at similar numbers were made within the scope of the construction aspect (13) in the cognitive process category and the sequential perception aspect (7) in the perceptual category shows that there may be a correlation between these two aspects. It is seen that there are no aspects that can be evaluated in the cognitive apprehension category at similar numbers to the reasoning aspect in the cognitive perception category. The diagram shows that the discursive apprehension aspect, which has the least representation throughout the course observed, is independent of the others. Thus, no findings encountered in C1's class can be evaluated in the operative apprehension aspect.

Table 5. *Dialogue Map in the Context of the Reasoning Processes of T1's Classroom Practices*

Section	Time interval	Cognitive processes	Cognitive apprehensions
Section1	0.50-4.15	4-1-5(A)	Perceptual
Section2	4.15-12.45	5(A)-3-4-5(B)	sequential–perceptual –discursive
Section3	12.48-16.19	2-5(A)-5(B)-3	perceptual–discursive
Section4	16.50-23.57	4	sequential-perceptual
Section5	24.24-40.20	2-5(B)	perceptual-discursive

Table 5 shows that T1's classroom practice was divided into five significant sections and that the cognitive process and cognitive apprehension categories constitute the dialog map through the data obtained from these sections. Looking closely at the table, various dialog maps exist in each section. When Table 3 is analyzed, it is seen that behaviors can be evaluated within the scope of the perceptual apprehension in the cognitive apprehension category in each section, simultaneously with the cognitive process aspects. When the geometrical activity of the teacher in sections 1, 2, 3 and 5 is considered, it is possible to see a rich interaction between the aspects in the cognitive process category. In contrast, a unidirectional interaction is observed in section 4.

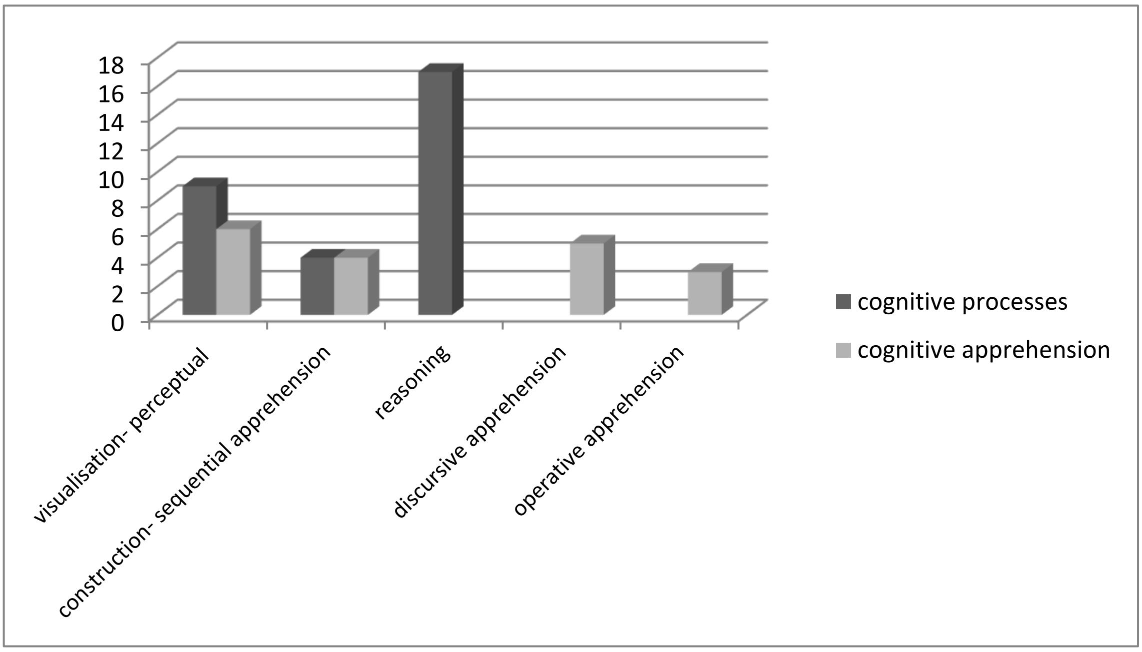
The teacher asks questions to the students to gather their attention, test their preliminary knowledge, and start the new subject through visuals before starting the course in section 1. For instance, it was observed that the teacher asked the students what a triangle was or why it was called a triangle before starting the subject of triangle inequality. The students attempted to create a triangle by describing the construction of a triangle. Then, it was observed that the teacher brought meaning to the features of the triangle visual shown on the smartboard. In section 2, the teacher told the students to draw a triangle having side lengths of 1-3-5 cm using a ruler.

Nevertheless, the students initially attempted to draw this triangle without following any sequences or directions and stated that they failed. When the teacher asked the students why they were having this much difficulty, one student expressed that they needed specific rules for this drawing. The teacher tried to make the students find the rule at this point. In sections 3 and 5, however, the teacher included the students in the process and asked questions to get them to make inferences to bring meaning to the subject. Besides, it was observed that the teacher guided the students at specific points that could have been needed to solve the problems during the practices implemented to get the students to solve exercises on the smartboard. Nonetheless, the teacher acted as a lecturer, while the students were listeners in section 4. Moreover, the results included the behaviors that could be evaluated in the reasoning aspect of the cognitive process category in every section, excluding section 4.

CLASSROOM PRACTICES OF T2 IN THE CONTEXT OF GEOMETRIC REASONING PROCESSES

The aspects obtained from the analysis of the geometrical reasoning process of T2's class are given in Diagram 2.

Diagram 2. *Geometric Reasoning Processes and Frequencies in T2's Classroom Practices*



Based on Diagram 2, when T2's classroom practices are evaluated within the context of the geometrical reasoning processes, it was observed that the visualization aspect in the cognitive process category was emphasized nine times. The perceptual apprehension aspect in the cognitive apprehension category was emphasized six times. Furthermore, evaluations at the same numbers were made within the scope of the construction aspect (4) in the cognitive process category, and the sequential perception aspect (4) in the perceptual category shows a strong connection between these two aspects. The reasoning aspect was emphasized the most in the cognitive perception category.

Table 6. *Dialogue Map in the Context of the Reasoning Processes of T2's Classroom Practices*

Section	Time interval	Cognitive processes	Cognitive apprehensions
Section1	00.00-03.25	5(B), 5(A)-3-4	discursive – Perceptual – sequential
Section2	3.23-7.26	2-5(B)-3	sequential – Perceptual – discursive
Section3	7.56-11.30	2-5(B), 5(A)	Perceptual – discursive
Section4	11.30-18.50	2- 5(B)	Perceptual –operative
Section5	18.50-25.20	2-5(A)-3	sequential – Perceptual – discursive
Section 6	25.20-27.30	2-5(B), 5(A)-3	sequential – Perceptual-discursive
Section7	32.40-42.00	5(B)-1	discursive –operative

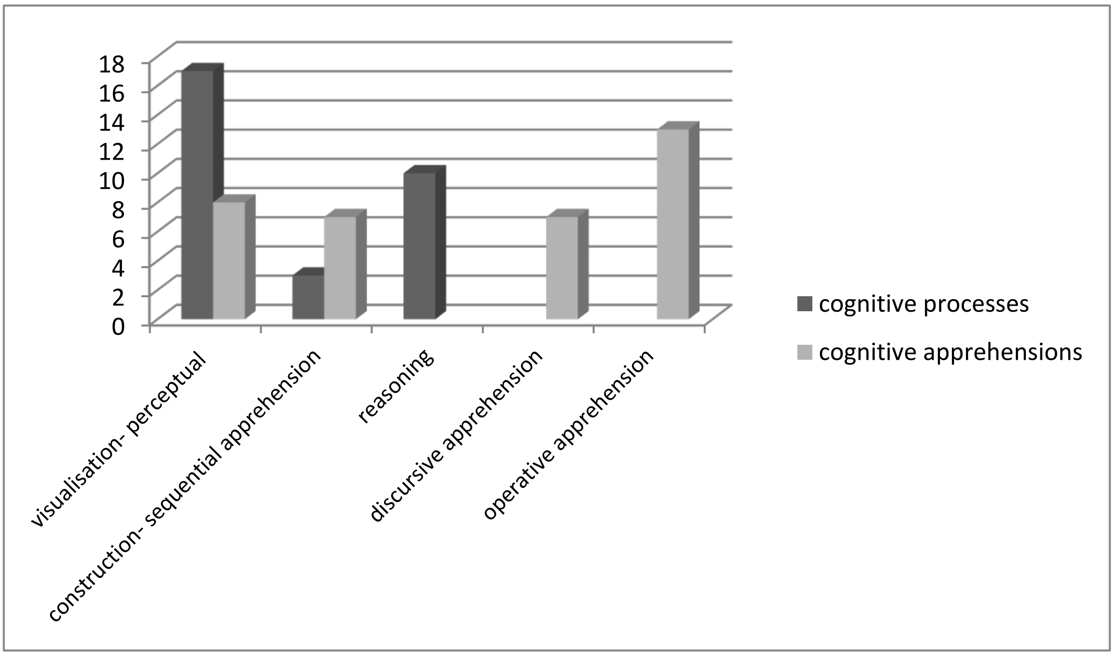
Table 6 illustrates that T2's classroom practice was divided into seven significant sections. The cognitive process and cognitive apprehension categories constitute the dialog map through the data obtained from these sections. Some features can be evaluated within each section's scope of the discursive apprehension in the cognitive apprehension category, simultaneously with the cognitive process aspects. Furthermore, it is also observed that interactions can be evaluated within the scope of the perceptual apprehension in the cognitive apprehension category in each section, excluding section 7, simultaneously with the cognitive process aspects.

In the T2 classroom practices, rich interactions are observed between the aspects in the cognitive process category, between the aspects in the cognitive apprehension category, and between the cognitive process category and cognitive apprehension category. Furthermore, T2 followed the direction from visualization to reasoning in the cognitive process category in each section, excluding sections 1 and 7. T2 kept students' attention alive by asking questions aimed at change and expansion in the students' knowledge at every stage of the course. For instance, it was observed that the teacher invited one student to the board to draw a square and asked other students questions, such as "Is this a square? Why not? What does it need to be a square?" as this student kept drawing, to make them following a specific sequence and give correct answers.

CLASSROOM PRACTICES OF T3 IN THE CONTEXT OF GEOMETRIC REASONING PROCESSES

The aspects obtained from the analysis of the geometrical reasoning process of T3's classroom are given in Diagram 3.

Diagram 3. *Geometric Reasoning Processes and Frequencies in T3's Classroom Practices*



Based on Diagram 3, T3 mostly used the visualization aspect (17) and the construction aspect the least (3) concerning the cognitive perception process. Moreover, it is seen that the operative apprehension aspect (15) was used the most concerning the perceptual apprehension process.

Table 7. *Dialogue Map in the Context of the Reasoning Processes of T3's Classroom Practices*

Section	Time interval	Cognitive processes	Cognitive apprehensions
Section1	00.00-06.30	2 - 5(A) - 5(B) - 3	Perceptual - discursive – sequential
Section2	07.13-08.05	2 - 5(A) - 3	Perceptual - discursive – sequential
Section3	08.05-08.55	2 - 5(A) - 3	Perceptual – discursive – sequential
Section4	9.50-11.00	2 - 5(A)	Discursive - Perceptual
Section5	11.00-16.50	2 - 5(B)	Perceptual – discursive
Section 6	16.50-20.52	2-5(B)	Perceptual – operative
Section7	20.52-24.40	2-5(A),5(B)-3	Perceptual - discursive - sequential –operative
Section8	24.40-30.00	2-5(A)	Perceptual - discursive – operative
Section9	30.00-33.50	2-5(A),5(B)	Perceptual - operative – discursive
Section10	33.50-35.25	2-5(A)	Perceptual - operative –discursive
Section11	35.30-40.00	2-5(A)	Perceptual - operative –discursive

When Table 7 is analyzed, T3's classroom practice was divided into 11 significant sections, and the cognitive process and cognitive apprehension categories constitute the dialog map through the data obtained from these sections.

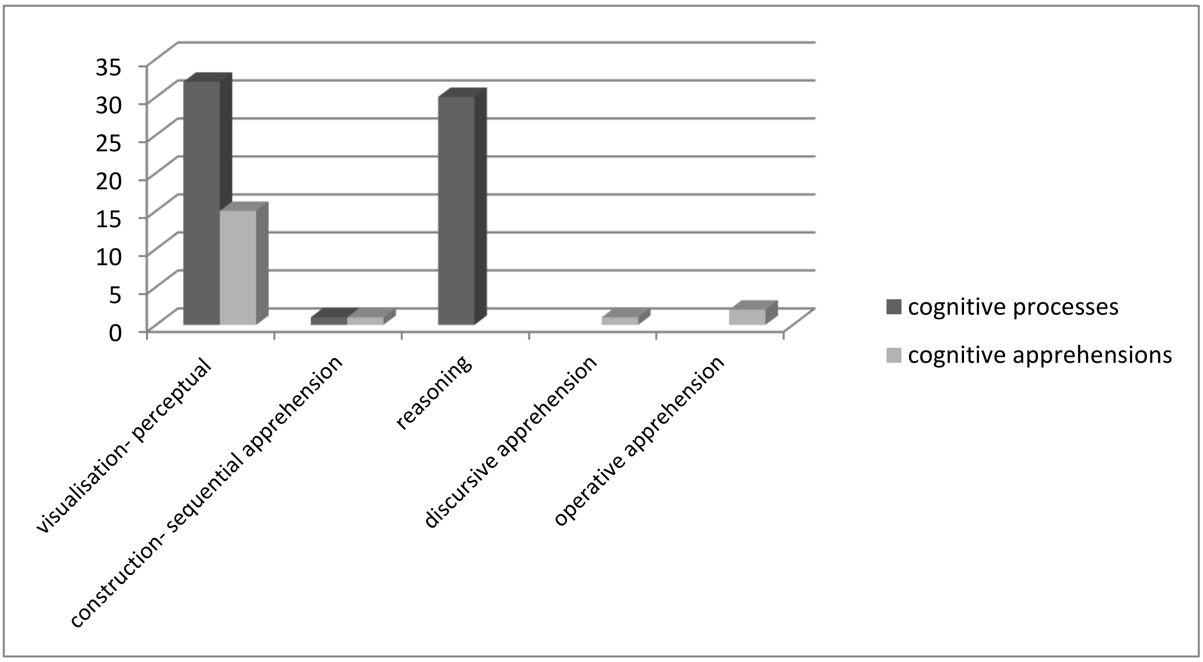
There are similar cognitive processes and cognitive apprehensions between sections 1, 2 and 3, 4 and 5, and 8, 9, 10, and 11. In section 7, however, it can be seen that a chain is created with a rich interaction in the form of 2-5(A), 5(B) -3, within the context of the cognitive process. A geometrical activity was obtained by experiencing a perceptual apprehension process intertwined with the natural discursive process and the discursive apprehension and operative apprehension processes intertwined with the theoretical process. Besides, it was determined that there were behaviors that could be evaluated within the scope of the perceptual apprehension in the cognitive apprehension category in each section, simultaneously with the cognitive process aspects. Furthermore, behaviors can be evaluated within the scope of the discursive apprehension in the cognitive apprehension category in each section, excluding section 6, simultaneously with the cognitive process aspects. As of section 6, behaviors that could be evaluated in the operative apprehension aspect in the cognitive apprehension category were observed.

In each section of T3's classroom practice, particularly in section 7, rich interactions are observed between the cognitive process category, the aspects in the cognitive apprehension category, and the cognitive process category and cognitive apprehension category. Furthermore, the teacher followed the direction from visualization to reasoning in the cognitive process category in each section. During the class observation, T3 directed the students to solve exercises to let them see the different aspects of the shapes. For instance, the teacher attempted to make it easier for the students to see the corresponding angles, alternate interior angles, and alternate exterior angles between two parallel lines by drawing auxiliary straight lines for both lines or extending them.

CLASSROOM PRACTICES OF T4 IN THE CONTEXT OF GEOMETRIC REASONING PROCESSES

The aspects obtained from the analysis of the geometrical reasoning process of T4 are given in Diagram 4.

Diagram 4. *Geometric Reasoning Processes and Frequencies in T4's Classroom Practices*



In diagram 4, T4 used the visualization (32) and reasoning (28) aspects within the cognitive processes context while using the construction aspect only once. However, in the cognitive apprehension context, the perceptual apprehension aspect (15) became prominent, while only a little emphasis was put on other aspects.

Table 8. *Dialogue Map in the Context of the Reasoning Processes of T4's Classroom Practices*

Section	Time interval	Cognitive processes	Cognitive apprehensions
Section1	00.00-06.00	2-5(A),5(B)-3	Perceptual - discursive – sequential
Section2	06.00-10.00	2-5(A),5(B)-3	Perceptual - discursive – sequential
Section3	10.00-14.15	2-5(B), 5(A)-3	Perceptual–operative- sequential
Section4	18.50-23.53	2- 5(A),5(B)	Perceptual – discursive
Section5	23.53-26.50	2-5(A)	Perceptual
Section 6	34.50-40.00	2-5(A)	Perceptual

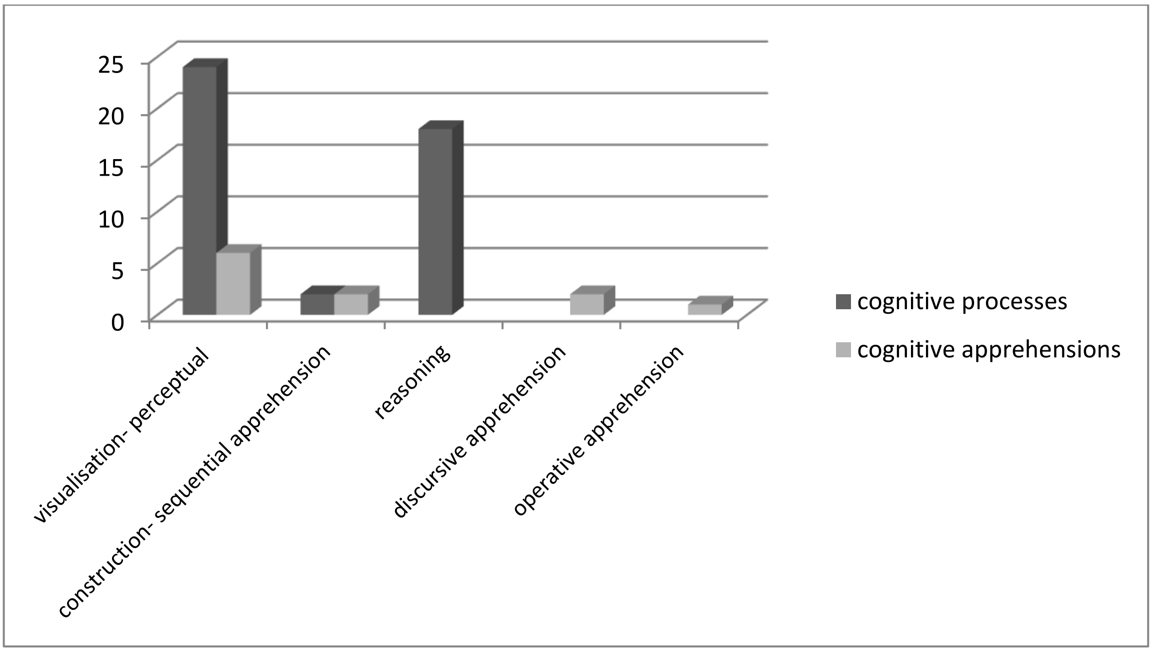
Table 8 shows that T4's classroom practice was divided into six significant sections based on the analysis. It can be seen that the perceptual apprehension aspect in the cognitive apprehension category became prominent in each section, simultaneously with the cognitive process aspects. The operative apprehension aspect in the cognitive apprehension category became prominent only in section 3, simultaneously with the cognitive process aspects.

T4 followed the direction from visualization to reasoning in the cognitive process category in each section. The teacher aimed to ensure that the students acquired new information about concrete objects during the class. For instance, the teacher showed the concepts of an acute, right, obtuse angle, using the hour and minute hands of a wall clock in the classroom and the visuals on the smartboard.

CLASSROOM PRACTICES OF T5 IN THE CONTEXT OF GEOMETRIC REASONING PROCESSES

The aspects obtained from the analysis of the geometrical reasoning process of T5 are given in Diagram 5.

Diagram 5. *Geometric Reasoning Processes and Frequencies in T5's Classroom Practices*



When Diagram 5 was analyzed, T5 used the visualization aspect (23) the most. Furthermore, it can also be seen that, out of the geometrical reasoning aspects, T5 used the operative apprehension aspect the least.

T5 used the visualization (24) and reasoning (18) aspects within the cognitive processes context while using the construction aspect only twice in diagram 5. However, in the cognitive apprehension context, the perceptual apprehension aspect (6) became prominent, while only a little emphasis was put on other aspects.

Table 9. *Dialogue Map in the Context of the Reasoning Processes of T5's Classroom Practices*

Section	Time interval	Cognitive processes	Cognitive apprehensions
Section1	00.00-02.38	2-5(A)-3	Perceptual - discursive – sequential
Section2	02.38-05.05	2-5(A)	Perceptual
Section3	05.05-07.07	2-5(A)	Perceptual
Section4	09.30-12.01	2-5(A)	Perceptual
Section5	12.01-14.40	2-5(A),5(B)	Perceptual
Section 6	14.40-16.20	2-5(A)	Perceptual
Section7	16.20-17.10	2-5(A)	Perceptual- operative
Section8	35.35-36.40	2-5(A)	Perceptual

When looking at Table 9, it is clear that T5's classroom practice was divided into eight significant sections. Out of these sections, sections 2, 3, 4, 5, 6, and 8 consisted of similar dialog maps. A chain was created in the form of 2-5(A)-3 in the cognitive process category of section 1. It can be seen that the perceptual apprehension behavior in the cognitive apprehension category was present in each section, simultaneously with the cognitive process aspects. Moreover, in-class activities can be evaluated within the scope of the operative apprehension in the cognitive apprehension category only in section 7, simultaneously with the cognitive process aspects. For instance, the teacher asked the students to think of the pencil's point as infinite or the straight line's ends as infinite. This example can be considered within the scope of the operative apprehension as the teacher followed the direction from visualization to reasoning in the cognitive process category in each section, excluding sections 1 and 7.

DISCUSSION, CONCLUSION, AND IMPLICATIONS

This study was examined within the scope of geometric reasoning processes of geometry-based mathematics courses. First, the findings on the frequency of use of geometric reasoning processes are discussed. Then, the findings on the interaction of cognitive and perceptual processes among themselves were discussed.

DISCUSSION ABOUT THE FREQUENCIES OF USE OF THE GEOMETRICAL REASONING PROCESSES

As per the results obtained from the study, the most prominent cognitive process aspect was the visualization in other classes, except for that of T2. Arcavi (2003) deems perceptual and conceptual reasoning the key element of visualization. The fact that the teachers emphasized the visualization aspect the most out of all geometrical reasoning processes during their classes shows that they emphasize the necessity that space should be visually represented for them to carry out functions, such as the visual demonstration of a situation, a general overview of the current situation, instantaneous perceptions, and subjective verification, for their students (Duval, 1995). Visualization is a method used by teachers since it functions as a strong tool for bringing meaning to mathematical concepts and associating them as it enables the abstract space to become concrete, as well as for structuring the knowledge, for decreasing the complexity when dealing with multiple information, for facilitating the solution of mathematical problems, thus for constituting a basis for abstract thinking, for applying mathematics on daily life, and for making the students love mathematics. Therefore, Duval (1999) claims that symbolic expressions and visualization are essential for understanding mathematics.

Based on the study's findings, the teachers' strong emphasis on the reasoning process can be evaluated as a positive issue. Geometrical thinking also stands for reasoning. Some researchers (Diezmann & English, 2001; English, 1998; Kramarski & Mizrachi, 2004; Kramarski et al., 2001; Curtis, 2004; Schoenfeld, 1992; Sparkes, 1999; Toole, 2001; White, Alexander & Daugherty, 1998) also support the fact that reasoning is significant for the teaching of mathematics (geometry) effectively.

Based on the results obtained from the study, it was seen that the visualization aspect in the cognitive process category and the perceptual apprehension aspect in the cognitive apprehension category support one another. Then again, it was also seen that the construction aspect in the cognitive process category and the sequential perception aspect in the cognitive apprehension category support one another in all teachers. Nevertheless, it was observed that the construction code did not become quite prominent, except for T1's class. Geometrical constructions are important in teaching geometry meaningfully (Martin, 2012).

DISCUSSION OF THE RESULTS REGARDING THE INTERACTION OF COGNITIVE PROCESSES AND COGNITIVE APPREHENSIONS

As a result of the analysis of the teacher-student roles as well as the dialogue maps in the departments with rich interactions in the context of both cognitive processes and cognitive apprehensions, as well as the observational results obtained from the research, students are active. The teacher plays a guiding role while the student takes active responsibility in learning. In the sections where limited interactions were observed, it was determined that the teachers conveyed information and the students played the role of passive listeners. Accordingly, various studies have stated that in learning environments with rich and superior interaction, active participation of students in the learning process will provide opportunities for them to make sense of information rather than memorize it (Eriksson, Helenius & Ryve, 2019; King, 1993; Mierson & Parikh, 2000).

It was observed that cognitive processes were emphasized naturally in three of the classes examined (T3, T4, and T5). In one (T2) the theoretical discursive process was emphasized, and in one (T1), these processes were mixed. In the context of cognitive apprehensions only the visual dimension is used more. In two classes (T1 and T4), visual perception and sequential perception aspects are frequently used simultaneously, and in two classes (T2 and T3), more than two dimensions are mainly used. This shows that geometric reasoning processes differ from class to class. When analyzed in this respect, there may be some differences regarding the development of students taking courses from different teachers in geometric reasoning processes. However, this may lead to advantageous or disadvantageous factors for the relevant geometric reasoning styles of students going through the same educational processes. From this point of view, it appears that the teaching processes of teachers should be examined more meticulously.

The emphasis on other aspects was less than the visualization and reasoning aspects in the cognitive process, and cognitive apprehension categories may be stemming from the van Hiele effect over the curricula. According to the studies in the literature, secondary school students are at the first and second levels per the van Hiele geometrical Thinking levels (Nisawa, 2018). Based on these levels, students cannot make upper-level inferences, such as theorems and axioms, as they are visual and descriptive. These skills initially surface when the visual and reasoning levels are improved as it is believed that it is not possible to reach superior skills, such as operative apprehension and discursive apprehension, in Duval's cognitive model.

The results obtained from the dialog maps put forward that there was no hierarchical structure between the aspects in the geometrical reasoning process, but rather an interactive, simultaneous, and sometimes independent one was present. The fact that these aspects were interactive will ensure effective learning while being independent enables each aspect to be improved individually. Nevertheless, it can be stated that the evaluation of these actual results, which are limited to the preferences of the participating teachers, by focusing on different intra-class practices from a higher number of teachers can be a significant endeavor to achieve generalizable results. Then again, it is also considered that the studies, which would cover different geometrical subjects and unit-based evaluations to set forth the effectiveness of Duval's model in particular, would enable a wider spectrum for the discussions that constitute the theoretical foundation of this study (Duval, 1999; Fischbein, 1993; Herbst, 2006; Hoffer, 1981; Piaget & Inhelder, 1967; van Hiele, 1957).

Also, it is prominent that the participants' teaching processes differed in geometrical reasoning processes. The results obtained at the end of the study showed that the teachers emphasized the visualization aspect in the cognitive process category in their classes. However, the second emphasis was put on the reasoning aspect in the same category.

Geometry teaching depends on how well the teachers know geometry and how effective they can teach it (Jones, 2000; Sunzuma & Maharaj, 2019). Except for the methodical and theoretical recommendations, there may also be several practice-oriented perspective recommendations in light of the results obtained from this study. Accordingly, it may be possible to offer occupational development opportunities aimed at a student-oriented mathematics education by preparing examples that would take Duval's (1999) model as a basis, and that would take the interaction of the cognitive and perceptual aspects of the teacher-student roles in the intra-class practices of the teachers to a higher level as much as possible.

AUTHOR CONTRIBUTION

-First author have been involved in drafting the manuscript or revising it critically for important intellectual content.

-The second author have been involved revising it critically for important intellectual content.

-The third author have made substantial contributions to design, or acquisition of data, or analysis and interpretation of data.

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